Neutrino Propagation Through Matter

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Abstract

We discuss a simple approach to solve the transport equation for high-energy neutrinos in media of any thickness. We present illustrative results obtained with some specific models for the initial spectra of ν_μ and $\overline{\nu}_\mu$ propagating through a normal cold medium.

Key words: Transport theory; Neutrino interactions

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1 Introduction

In passing through a medium, neutrinos (and antineutrinos) are absorbed and lose their energy due to charged and neutral current interactions. For a normal *cold* medium, like the earth's or stellar interior, these are νN ($\overline{\nu}N$) and νe ($\overline{\nu}e$) collisions. In more exotic media, like the relic black-body neutrino background [1] or *hot* galactic halos filled by massive neutrinos [2,3], ultrahigh energy neutrinos may scatter elastically or be absorbed due to $\nu\overline{\nu}$ interactions. Owing to the energy loss and the strong energy dependence of total cross sections for neutrino interactions, the neutrino "depth-intensity relation" (or penetration coefficient [4]) does not follow a simple absorption law and the magnitude of this effect grows with energy and depth. Such a situation is well known in the muon transport theory (see e.g. ref. [5]). To solve the neutrino transport equation for moderate depths, the method of successive generations is workable [6]. But this method becomes inefficient for the depth in excess of several neutrino interaction lengths. Pertinent refining of the neutrino transport theory is desirable for many applications, specifically for studying standard and speculative neutrino interactions (see e.g. ref. [7] and references therein), detecting neutrino signals from annihilation of dark matter particles in the sun and the earth, and for high-energy neutrino astronomy with future, km^3-scale neutrino telescopes [8,9].

The goal of this work is to provide an elementary method for the precise calculation of the energy spectra of high-energy neutrinos after their propagation through a medium of *any* thickness. The problem was considered recently in ref. [10] in the framework of a simplified models for the neutrino cross sections and initial neutrino spectrum. Our approach does not require simplifications and is applicable to cross sections (differential and total) and initial spectra of any form. In sections 2 and 3, we shall consider only decreasing unbroken initial spectra most interesting for high-energy neutrino astrophysics. However, the main idea of the method can be extended also to a monochromatic spectrum (see appendix A). This generalization may be of utility, in particular, for simulating single neutrino events in a neutrino telescope. Furthermore, the method makes no assumptions specific to neutrino transport and can be extended almost straightforwardly to the problem of transport of high-energy particles other than neutrinos (e.g. hadrons and muons).

Under certain conditions, neutrinos may transform, changing their flavor via processes like $\overline{\nu}_e e^- \to \overline{\nu}_\ell \ell^-$ or $\nu_\ell e^- \to \nu_e \ell^-$ ($\ell \neq e$), owing to production and decay of short-lived hadrons (D, D_s, etc.), or (in the mentioned neutrino fields) through reaction chains like $\nu_\mu \overline{\nu}_\tau \to \mu^- \tau^+$, $\tau^+ \to \overline{\nu}_\tau X$, etc. As it pointed out recently [11], high-energy tau neutrinos (and antineutrinos) will effectively regenerate in matter, losing energy, through the charged-current reaction chain $\nu_\tau N \to \tau X$, $\tau \to \nu_\tau X$. Such mechanisms must be taken into account in data processing from many future experiments (detecting ν_τ events from astrophysical neutrino oscillations at energies $\gtrsim 1$ PeV [12], events with energy release well beyond the Greisen-Zatsepin-Kuz'min cutoff [3], etc.).

Mathematically, the inclusion of processes changing the neutrino flavor or neutrino energy loss through creation and decay of short-lived particles leads to a system of transport equations. The extension of the method to the general system is not straightforward and demands additional assumptions specific to the task. However, the case when these contributions may be treated as corrections presents no special problem. Since this case is rather common (the neutrino production in the ν-induced hadronic cascades is a typical example), we brief the corresponding trivial generalization in appendix B.

To avoid technical complications, in the main text we shall neglect the (standard and hypothetical) flavor-changing neutrino interactions[1] and use the simplest "standard" scenario for neutrino propagation described by a *single* transport equation. We shall consider sufficiently high energies in order to neglect the thermal velocities of the scatterers in the target medium and to deal with the one-dimensional theory. As an illustration, we shall discuss results obtained with some specific models for the initial spectra of muon neutrinos and antineutrinos propagating through a normal cold medium.

2 Method for the solution of the neutrino transport equation

Let $F_\nu(E,x)$ be the differential energy spectrum of neutrinos at a column depth x in the medium defined by

$$x = \int_0^L \rho(L')dL',$$

where $\rho(L)$ is the density of the medium at a distance L from the boundary measured along the neutrino beam path. Then, neglecting the flavor-changing processes mentioned in the introduction, one can derive the following one-dimensional transport equation

$$\frac{\partial F_\nu(E,x)}{\partial x} = \frac{1}{\lambda_\nu(E)}\left[\int_0^1 \Phi_\nu(y,E)F_\nu\left(\frac{E}{1-y},x\right)\frac{\mathrm{d}y}{1-y} - F_\nu(E,x)\right], \quad (1)$$

[1] As well as the effects of possible neutrino flavor mixing.

with the boundary condition $F_\nu(E,0) = F_\nu^0(E)$. Here $\lambda_\nu(E)$ is the neutrino interaction length defined by the equation

$$\frac{1}{\lambda_\nu(E)} = \sum_T N_T \sigma_{\nu T}^{\text{tot}}(E),$$

where N_T is the number of scatterers T in 1 g of the medium, $\sigma_{\nu T}^{\text{tot}}(E)$ is the total cross section for the νT interactions and the sum is over all scatterer types ($T = N, e, \ldots$). The "regeneration function" $\Phi_\nu(y,E)$ is defined by

$$\sum_T N_T \frac{\mathrm{d}\sigma_{\nu T\to\nu X}(y,E_y)}{\mathrm{d}y} = \Phi_\nu(y,E)\sum_T N_T \sigma_{\nu T}^{\text{tot}}(E),$$

where $\mathrm{d}\sigma_{\nu T\to\nu X}(y,E)/\,\mathrm{d}y$ is the differential cross section for the inclusive reaction $\nu T \to \nu X$ (with E the initial neutrino energy and y the fraction of energy lost) and $E_y \equiv E/(1-y)$.

Let us define the effective absorption length $\Lambda_\nu(E,x)$ by

$$F_\nu(E,x) = F_\nu^0(E)\exp\left[-\frac{x}{\Lambda_\nu(E,x)}\right]. \tag{2}$$

As is evident from eq. (1), $\Lambda_\nu(E,x) > \lambda_\nu(E)$ for any finite E and x. Therefore

$$\Lambda_\nu(E,x) = \frac{\lambda_\nu(E)}{1 - Z_\nu(E,x)}, \tag{3}$$

where $Z_\nu(E,x)$ is a positive function (we will call it Z factor in analogy with the hadronic cascade theory) which contains the complete information about the neutrino kinetics in matter.

Substituting eqs. (2) and (3) into eq. (1) and integrating by parts, it is easy to derive the integral equation for the Z factor:

$$Z_\nu(E,x) = \frac{1}{x}\int_0^x\int_0^1 \eta_\nu(y,E)\Phi_\nu(y,E)\exp\left[-x' D_\nu(E,E_y,x')\right]\,\mathrm{d}x'\,\mathrm{d}y, \tag{4}$$

with

$$D_\nu(E,E_y,x) = \frac{1 - Z_\nu(E_y,x)}{\lambda_\nu(E_y)} - \frac{1 - Z_\nu(E,x)}{\lambda_\nu(E)}$$

and

$$\eta_\nu(y,E) = \frac{F_\nu^0(E_y)}{F_\nu^0(E)(1-y)}.$$

We dwell on eq. (4). Although nonlinear, it proves to be more amenable for an iteration solution than eq. (1), considering the smallness of the Z factor and (what is more important) a model-independent feature of the regeneration function $\Phi_\nu(y,E)$, namely, its sharp maximum at $y=0$.

In this section, we assume that the initial spectrum $F_\nu^0(E)$ is a continuous function decreasing at high energies fast enough so that $0 \le \eta_\nu(y,E) < \infty$ for any E and $0 \le y \le 1$. Actually, the neutrino spectra of interest for high-energy neutrino astrophysics decrease everywhere so fast that $0 \le \eta_\nu(y,E) < 1$ for any E and $y > 0$.

We will first look at the case of thin absorbers. One can readily see that

$$Z_\nu(E,0) = \int_0^1 \eta_\nu(y,E)\Phi_\nu(y,E)\,\mathrm{d}y \equiv Z_\nu^0(E).$$

The approximation $Z_\nu(E,x) = Z_\nu^0(E)$ is usually utilized when studying the muon neutrino propagation through matter (see e.g. ref. [4,8] and references therein). However, at high energies this approximation becomes too rough even for "shallow" (as compared to λ_ν) depths. Indeed, taking into account the $\mathcal{O}(x/\lambda_\nu)$ correction yields

$$Z_\nu(E,x) \approx Z_\nu^0(E) - \frac{x\Delta_\nu^1(E)}{2\lambda_\nu(E)},$$

where

$$\begin{aligned}
\Delta_\nu^1(E) &= -\lambda_\nu(E)\left[\frac{\partial Z_\nu(E,x)}{\partial x}\right]_{x=0} \\
&= \int_0^1 \eta_\nu(y,E)\Phi_\nu(y,E)\left\{\left[1-Z_\nu^0(E_y)\right]\frac{\lambda_\nu(E)}{\lambda_\nu(E_y)} - \left[1-Z_\nu^0(E)\right]\right\}\mathrm{d}y.
\end{aligned}$$

Thus, the approximation $Z_\nu \approx Z_\nu^0$ can only be valid for

$$\frac{x}{\lambda_\nu(E)} \ll \frac{2Z_\nu^0(E)}{|\Delta_\nu^1(E)|},$$

In the general case, the function $\Delta_\nu^1(E)$ is not small in comparison with $Z_\nu^0(E)$. This can be demonstrated with the simple model adopted in ref. [10]. The authors of ref. [10] assumed that $\Phi_\nu = \Phi_\nu(y)$ is an energy independent function, $\lambda_\nu(E) \propto E^{-\beta}$ and $F_\nu^0(E) \propto E^{-(\gamma+1)}$ with energy independent positive β and γ. All these assumptions are far from reality but may have a physical sense at super-high energies. For example, owing to the νN interactions, β is a monotonically decreasing function of E changing from about 1 at $E \lesssim 1$ TeV to about 0.4 at $E \gtrsim 1$ PeV [13]; the function $\Phi_\nu(y, E)$ strongly varies with E at all energies, but for $E \gtrsim 1$ PeV it may be roughly approximated by a scaling function (see fig. 1 in sect. 3).[2] In this model, both Z_ν^0 and Δ_ν^1 are energy independent:

$$Z_\nu^0 = \int_0^1 (1-y)^\gamma \Phi_\nu(y)\,\mathrm{d}y,$$

$$\Delta_\nu^1 = \left(1 - Z_\nu^0\right) \int_0^1 (1-y)^\gamma \left[(1-y)^{-\beta} - 1\right] \Phi_\nu(y)\,\mathrm{d}y.$$

Evidently Δ_ν^1 can be much larger than Z_ν^0 for a sufficiently hard initial neutrino spectrum (small γ)[3].

It is not a hard task to derive the $\mathcal{O}\left((x/\lambda_\nu)^k\right)$ corrections for $k = 2, 3, \ldots$, but as a result we will get an asymptotic expansion with coefficient functions, $\Delta_\nu^k(E)$, increasing fast with k. The region of applicability of this expansion proves to be very limited and decreases fast with increasing energy.

Now, let us consider a way to solve eq. (4) for any depth and energy. We will use an iteration algorithm. Let n label the iteration number. Then we define

$$D_\nu^{(n)}(E, E_y, x) = \frac{1 - Z_\nu^{(n)}(E_y, x)}{\lambda_\nu(E_y)} - \frac{1 - Z_\nu^{(n)}(E, x)}{\lambda_\nu(E)} \qquad (5)$$

and

$$Z_\nu^{(n+1)}(E, x) = \frac{1}{x} \int_0^x \int_0^1 \eta_\nu(y, E) \Phi_\nu(y, E) \exp\left[-x' D_\nu^{(n)}(E, E_y, x')\right] \mathrm{d}x'\,\mathrm{d}y. \qquad (6)$$

Due to the mentioned sharp maximum of $\Phi_\nu(y, E)$, the main contribution into the integral over y on the right side of eq. (6) comes from the lover limit

[2] However, as fig. 1.b suggests, the specific parametrization used in ref. [10], $\Phi_\nu(y) \propto (\mathrm{const} + y)^{-1}$, is too rough even at super-high energies.
[3] In the real case, $\Delta_\nu^1(E)$ is nevertheless finite because any physical spectrum $F_\nu^0(E)$ has a cutoff at some finite energy E_{cut} and therefore $\eta_\nu(y, E) = 0$ at $y \geq 1 - E/E_{\mathrm{cut}}$.

neighborhood. But $D_\nu(E,E_y,x) \to 0$ as $y \to 0$ and thus the algorithm is robust in respect to choosing the zero approximation. The simplest choice is $Z^{(0)}_\nu(E,x) = 0$. Therefore

$$D^{(0)}_\nu(E,E_y,x) = \frac{1}{\lambda_\nu(E_y)} - \frac{1}{\lambda_\nu(E)} \equiv \mathcal{D}_\nu(E,E_y), \tag{7}$$

The algorithm (5–7) is formally applicable for arbitrary decreasing initial spectra. It is however clear that the softer the initial spectrum, the better the convergence of the algorithm. In the next section, we show that the algorithm converges very fast for realistic initial spectra and has no restrictions in depth or energy. Moreover, even the first approximation,

$$Z^{(1)}_\nu(E,x) = \int_0^1 \eta_\nu(y,E)\Phi_\nu(y,E)\left[\frac{1-\mathrm{e}^{-x\mathcal{D}_\nu(E,E_y)}}{x\mathcal{D}_\nu(E,E_y)}\right] \mathrm{d}y, \tag{8}$$

proves to be quite accurate. It has the correct asymptotic behavior both in energy and depth and can thus be used for an analytical or numerical evaluation of the Z factor with a not-too-big error.

Assuming that $\lambda_\nu(E)$ is a decreasing function [4] and therefore $\mathcal{D}_\nu(E,E_y) > 0$ for $y > 0$, one can prove that

- $Z^{(1)}_\nu(E,x) \leq Z^0_\nu(E)$ at any E and $x \geq 0$, and
- $Z^{(1)}_\nu(E,x) \to 0$ as $x \to \infty$ at any E.

The latter signifies that neutrino "regeneration" due to the inclusive reactions $\nu T \to \nu X$ becomes negligible at sufficiently large depths. This conclusion remains true for the exact solution of eq. (4), under rather general assumptions about the behavior of the initial neutrino spectrum and cross sections.

3 Numerical Illustration and Discussion

Below, we will consider only muon neutrinos and antineutrinos propagating through a normal medium. But, with obvious reservations, the results that follow also hold for electron neutrinos. In this calculation, we shall neglect neutrino scattering off electrons (thus we exclude electron antineutrinos from our consideration) as well as the neutrino "recreation" in the ν-induced cascades. For further simplification, we shall deal with an isoscalar medium neglecting nuclear effects.

[4] For a normal medium, this is true for all neutrino flavors except $\overline{\nu}_e$ (see ref. [13]).

To calculate the differential $\nu_\mu N$ and $\overline{\nu}_\mu N$ cross sections we use the approach of ref. [13] based on the renormalization-group-improved parton model and new experimental information about the quark structure of the nucleon (see appendix C). Various versions of different sets of parton density functions $q(\hat{x}, Q^2)$ are now collected in a large CERN program library PDFLIB [14]; they can be simply accessed by setting few parameters to choose the desired version. In this calculations, we selected, following ref. [13], the third version of the CTEQ collaboration model [15], because it is characterized by a particularly suitable extrapolation at very low Bjorken $\hat{x}$. The evolution in Q^2 is realized by next-to-leading order Altarelli-Parisi equations from initial $Q_0^2 = 2.56$ GeV^2.

The total cross sections for CC and NC inelastic scattering of muon neutrinos and antineutrinos off an isoscalar nucleon are shown in fig. 1.a as the solid (ν_μ) and dashed ($\overline{\nu}_\mu$) curves. Fig. 1.b shows the regeneration functions $\Phi_{\nu_\mu}(y, E)$ (solid curves) and $\Phi_{\overline{\nu}_\mu}(y, E)$ (dashed curves) versus y for several values of E (10^3 to 10^{12} GeV).

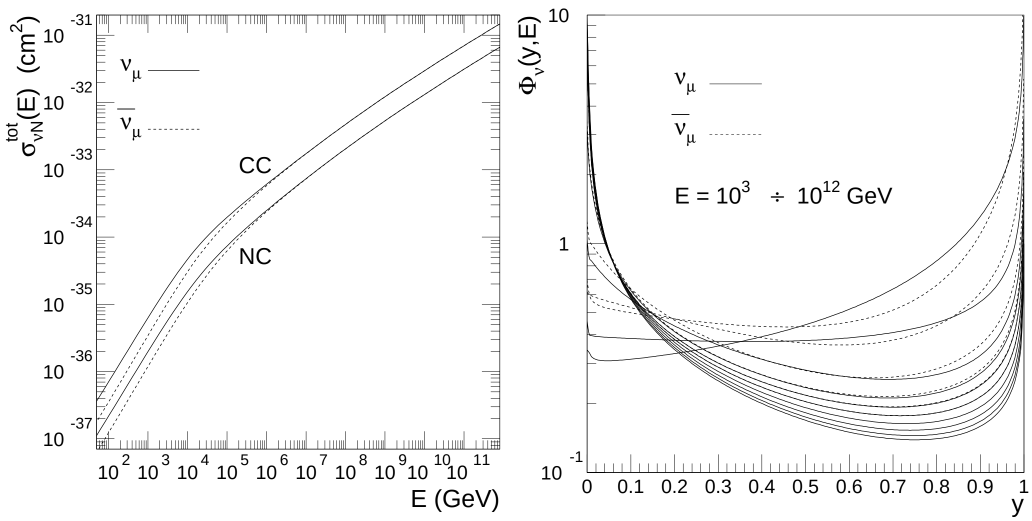

Fig. 1. Total (CC and NC) $\nu_\mu N$ and $\overline{\nu}_\mu N$ cross sections vs energy (a) and regeneration functions $\Phi_{\nu_\mu}(y, E)$ and $\Phi_{\overline{\nu}_\mu}(y, E)$ vs y for $E = 10^k$ GeV [$k = 3, 4, \ldots, 12$ from top to bottom] (b).

At all energies, our calculation for the cross sections agrees with the result of ref. [13] within a few percent accuracy; the insignificant difference near the resonance region is due mainly to differences in the adopted values for the electroweak parameters (W/Z boson masses, t quark mass, Weinberg angle, etc.) [5] and, at superhigh energies, to the top sea contribution neglected in ref. [13]. As one can see from the figures, the ν_μ and $\overline{\nu}_\mu$ scatterings become indistinguishable for $E \gtrsim 1$ PeV.

[5] In our calculation, all these parameters were updated according to the PDG data [16].

We use the following model for the initial neutrino spectrum:

$$F_\nu^0(E) = K\left(\frac{E_0}{E}\right)^{\gamma+1}\left(1+\frac{E}{E_0}\right)^{-\alpha}\phi\left(\frac{E}{E_{\text{cut}}}\right), \tag{9}$$

where K, γ, α, E_0 and E_{cut} are parameters and $\phi(t)$ is a function equal to 0 at $t \geq 1$ and 1 at $t \ll 1$. Varying the parameters in eq. (9), we can approximate many models for the neutrino fluxes expected from the known astrophysical sources. Technically, the function $\phi(t)$ serves to avoid an extrapolation of the cross sections to the ultrahigh energy region for which our knowledge of the parton density functions becomes doubtful. For realistic values of the parameters γ, α and E_0, the explicit form of $\phi(t)$ is of no importance if one is interested in the energy range $E \ll E_{\text{cut}}$. In fact, $\phi(t)$ may be treated as the real physical cutoff of the spectrum determined by the energetics of the neutrino source or by neutrino interactions with the cosmic backgrounds. In the present calculations, we adopt (without serious physics arguments) $\phi(t) = 1/\left[1+\tan\left(\pi t/2\right)\right]$ $(t<1)$ and $E_{\text{cut}} = 3\times 10^{10}$ GeV.

Fig. 2 shows the energy dependence of the Z factors, $Z_{\nu_\mu}(E,x)$ (solid curves) and $Z_{\overline{\nu}_\mu}(E,x)$ (dashed curves) for various depths, from $x=0$ to $x=x_\oplus$ (where $x_\oplus \approx 1.1\times 10^{10}$ g/cm^2 is the depth of the earth along the diameter), for the initial spectra (9) calculated with $\gamma=0.5$, $\alpha=1$ (a), $\gamma=1$, $\alpha=0.5$ (b), $\gamma=1.5$, $\alpha=0.5$ (c) and $\gamma=2$, $\alpha=1$ (d). In all cases we used $E_0 = 1$ PeV. The calculations were made in the fourth order of the iteration procedure described in sect. 2. For all the spectra under discussion, for 10 GeV $\leq E \leq 10^{10}$ GeV and $0 \leq x \leq x_\oplus$, the maximum difference between $Z_\nu^{(1)}(E,x)$ and $Z_\nu^{(2)}(E,x)$ is about 4%; the value $\left|Z_\nu^{(3)}/Z_\nu^{(2)}-1\right|$ is less than 2×10^{-3} and $\left|Z_\nu^{(4)}/Z_\nu^{(3)}-1\right|$ is less than the precision of the numerical integration and interpolation (about 10^{-5}) adopted in our calculations. After the tests with many models for the initial spectrum, we conclude that the convergence of the algorithm is very good and that even the first approximation, $Z_\nu^{(1)}(E,x)$, has an accuracy quite sufficient for the majority of applications of the theory.

As it is clear from fig. 2, the shape of the Z factors is very dependent from the initial spectrum. This is a positive fact for neutrino astronomy, since it gives, at least in principle, the possibility to reconstruct the initial neutrino spectrum from the measured energy spectrum and angular distribution of the neutrino induced muon events in a neutrino telescope.

At comparatively low energies (except for unrealistically hard spectra like the one used in fig. 2.a), the Z factors for antineutrinos exceed the ones for neutrinos. Considering the inequality $\lambda_{\overline{\nu}_\mu}(E) > \lambda_{\nu_\mu}(E)$, one can conclude that

$$\Lambda_{\overline{\nu}_\mu}(E,x) > \Lambda_{\nu_\mu}(E,x)$$

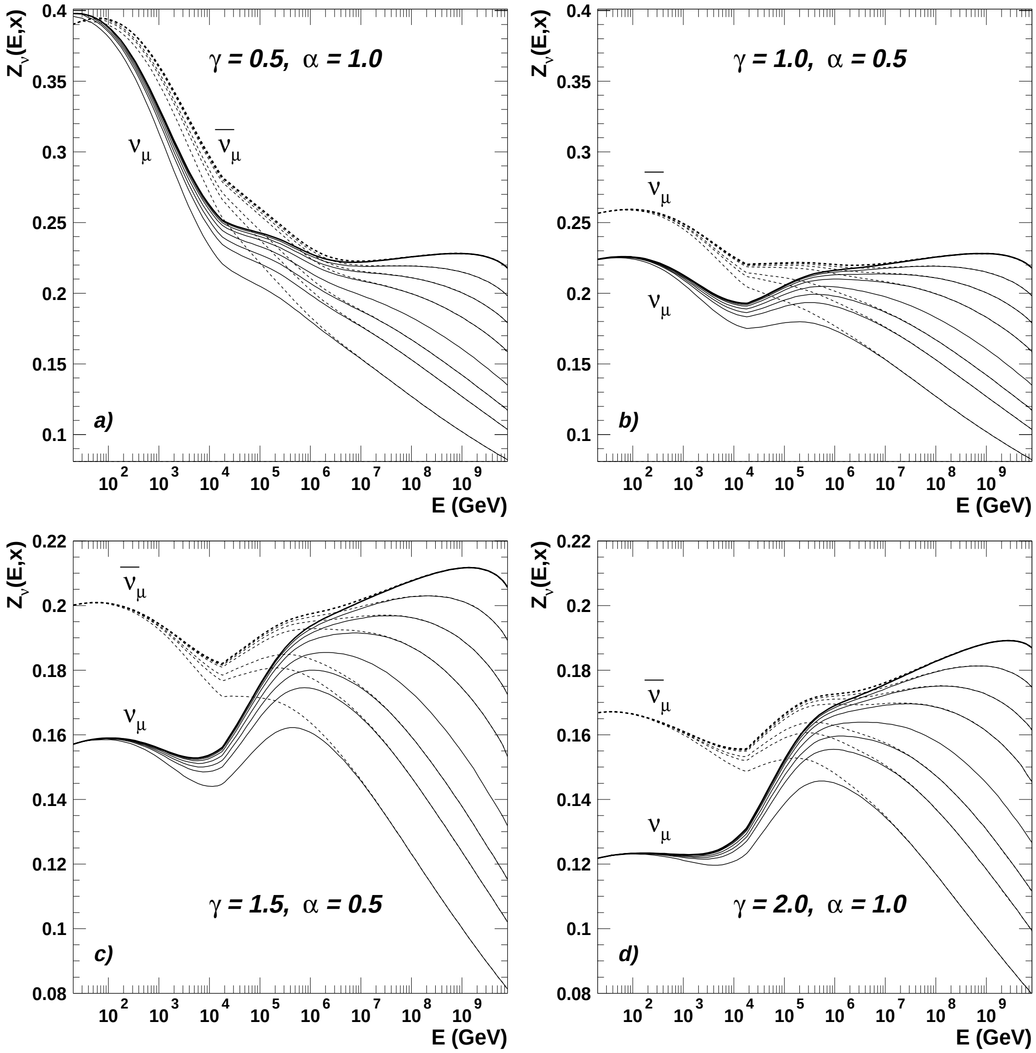

Fig. 2. Z factors, $Z_{\overline{\nu}_\mu}(E,x)$ and $Z_{\nu_\mu}(E,x)$ vs energy for the initial spectra (9), calculated with four different sets of γ and α and with $E_0 = 1$ PeV for depths $x = x_\oplus/k$ [$k = 1, 2, 3, 5, 10, 20, 50$ from bottom to top] and $x = 0$ (the largest Z factors).

for any depth. In the multi-PeV energy region and above, the Z factors (and effective absorbtion lengths) are identical for ν_μ and $\overline{\nu}_\mu$. The difference between the shapes of $Z_{\nu_\mu}(E,x)$ and $Z_{\overline{\nu}_\mu}(E,x)$ is almost depth-independent and becomes more important for steep initial spectra. This behavior may be understood from an analysis of the shapes of the total cross sections and regeneration functions for ν_μ and $\overline{\nu}_\mu$ (fig. 1).

At any fixed energy, the Z factors monotonically decrease with increasing depth and the inequality $Z_\nu(E,x) < Z_\nu^0(E)$ takes place for any $x > 0$. This effect leads to significant decrease of the neutrino event rates in comparison with those estimated in the "standard" approximation $Z_\nu \approx Z_\nu^0$; the latter works only at low energies, when the shadow effect is by itself small (that is when the medium is almost transparent for neutrinos). Although these conclusions were derived from particular models for the initial neutrino spectrum, cross sections, and medium, actually they are highly general and model-independent.

In fig. 3 we present the penetration coefficient, $\exp\left[-x/\Lambda_\nu(E,x)\right]$, in the earth for muon neutrinos with initial spectrum (9) calculated with $\gamma = 0.7$ and $\alpha = 0$ ("quasi-power-law" spectrum). The results are presented as a function of E for several nadir angles (ϑ) in fig. 3.a and as a function of ϑ for several values of E in fig. 3.b.

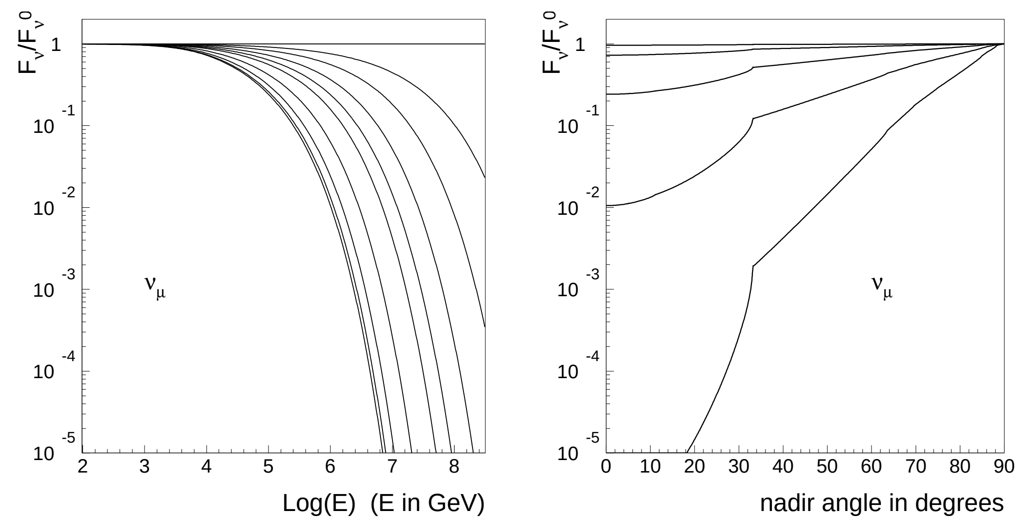

Fig. 3. Neutrino penetration coefficient in the earth for the quasi-power-law initial spectrum with $\gamma = 0.7$ as a function of E at fixed ϑ [0° to 90° from bottom to top with steps of 10°] (a) and as a function of ϑ for $E = 10^k$ GeV [$k = 3, 4, \ldots, 7$ from top to bottom] (b).

To evaluate the depth, x, as a function of ϑ, we used the density profile of the earth, $\rho(L)$, given in ref. [13]. The kinks in fig. 3.b are due to the layered structure of the earth.

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A Monochromatic initial spectrum

Let us consider an initial "spectrum" of the form $\delta(E-E_0)$ with a fixed parameter E_0. In this appendix, we will show how this monochromatic spectrum transforms at a depth x in a medium. Let us denote the transformed spectrum by $G_\nu(E_0;E,x)$. This function must satisfy eq. (1) and simple considerations suggest the following ansatz:

$$G_\nu(E_0;E,x) = \left[\delta(E-E_0) + \frac{\theta(E_0-E)}{E}\psi_\nu(E_0;E,x)\right] \mathrm{e}^{-x/\lambda_\nu(E_0)}, \quad \text{(A.1)}$$

where the term with δ function describes absorption of the initial ("parent") neutrinos of energy E_0 and the next term – the creation and propagation of secondary neutrinos with energy $E < E_0$. Substituting eq. (A.1) into eq. (1) yields

$$\begin{aligned} \frac{\partial \psi_\nu(E_0;E,x)}{\partial x} = & \frac{1}{\lambda_\nu(E)} \left[\int_0^{y_0} \Phi_\nu(y,E)\psi_\nu\left(E_0;E_y,x\right)\,\mathrm{d}y + \Omega_\nu\left(E,E_0\right)\right] \\ & + \mathcal{D}_\nu(E,E_0)\psi_\nu\left(E_0;E,x\right), \qquad \psi_\nu\left(E_0;E,0\right) = 0, \qquad \text{(A.2)} \end{aligned}$$

where $\Omega_\nu\left(E,E_0\right) = \left(1-y_0\right)\Phi_\nu\left(y_0,E\right)$, $y_0 \equiv 1 - E/E_0 < 1$, and $\mathcal{D}_\nu(E,E_0)$ is defined by eq. (7)

Let us seek the solution to eq. (A.2) in the form

$$\psi_\nu(E_0;E,x) = \Omega_\nu\left(E,E_0\right) \int_0^x \exp\left[\int_{x'}^x \frac{\mathrm{d}x''}{\mathcal{L}_\nu(E_0;E,x'')}\right] \frac{\mathrm{d}x'}{\lambda_\nu(E)}, \qquad \text{(A.3)}$$

$$\frac{1}{\mathcal{L}_\nu(E_0;E,x)} = \frac{1}{\lambda_\nu(E_0)} - \frac{1-\mathcal{Z}_\nu(E_0;E,x)}{\lambda_\nu(E)}, \qquad \text{(A.4)}$$

with $\mathcal{Z}_\nu(E_0;E,x)$ an unknown positive function. After direct substitution of eqs. (A.3) and (A.4) into eq. (A.2) we have

$$\mathcal{Z}_\nu(E_0;E,x)\psi_\nu(E_0;E,x) = \int_0^{y_0} \Phi_\nu(y,E)\psi_\nu\left(E_0;E_y,x\right)\,\mathrm{d}y. \qquad \text{(A.5)}$$

Clearly, $\mathcal{Z}_\nu(E_0;E,x) \to 0$ and $\psi_\nu(E_0;E,x) \to x\Phi_\nu(0,E_0)/\lambda_\nu(E_0)$ as $E \to E_0$ for any x.

The new "Z factor", $\mathcal{Z}_\nu(E_0; E, x)$, can be found from eqs. (A.3–A.5) by an iteration algorithm similar to the algorithm described in sect. 2. Putting $\mathcal{Z}_\nu = 0$ as a zero approximation we have

$$\psi_\nu^{(0)}(E_0; E, x) = \frac{\Omega_\nu\left(E, E_0\right)}{\lambda_\nu(E)\mathcal{D}_\nu(E, E_0)}\left[\mathrm{e}^{x\mathcal{D}_\nu(E,E_0)} - 1\right] \tag{A.6}$$

and subsequently

$$\mathcal{Z}_\nu^{(1)}(E_0; E, x) = \int\limits_0^{y_0} \Phi_\nu(y, E)\left[\frac{\psi_\nu^{(0)}\left(E_0; E_y, x\right)}{\psi_\nu^{(0)}\left(E_0; E, x\right)}\right] \mathrm{d}y. \tag{A.7}$$

The next steps of the algorithm are quite obvious so there is no need to write out the corresponding cumbersome formulas here.

Let us briefly sketch the leading approximations for ψ_ν and $\mathcal{Z}_\nu$, since they contain the main features of the exact solution. As it is seen from eq. (A.6), for every $E < E_0$ there is a depth

$$x_*(E_0, E) = \frac{1}{\mathcal{D}_\nu(E, E_0)} \ln \frac{\lambda_\nu(E)}{\lambda_\nu(E_0)}$$

at which the flux of neutrinos of energy E reaches the maximum. The function $x_*(E_0, E)$ increases when E decreases and tends to the minimum, $\lambda_\nu(E_0)$, as $E \to E_0$. At any finite depth, secondary neutrinos "remember" about their parents (the E_0 dependence does not disappear with increasing depth). Due to the nontrivial shape of the regeneration function Φ_ν (see fig. 1.b), the spectrum of secondary neutrinos is rather complex and transforms fast with depth.

For $x \ll \lambda(E_0)$, the function $\psi_\nu^{(0)}$ behaves as $x\Omega_\nu\left(E, E_0\right)/\lambda_\nu(E)$. Therefore

$$\mathcal{Z}_\nu^{(1)}(E_0; E, 0) = \int\limits_0^{y_0} \Phi_\nu(y, E)\left[\frac{\Omega_\nu\left(E_y, E_0\right)\lambda_\nu(E)}{\Omega_\nu\left(E, E_0\right)\lambda_\nu(E_y)}\right] \mathrm{d}y.$$

Taking into account that $\lambda_\nu(E) > \lambda_\nu(E_0)$ for $E < E_0$ (see footnote 4), we get the asymptotic behavior of $\mathcal{Z}_\nu^{(1)}$ for $x \to \infty$:

$$\begin{aligned}\mathcal{Z}_\nu^{(1)}(E_0; E, x) \sim & \int\limits_0^{y_0} \Phi_\nu(y, E)\left[\frac{\Omega_\nu\left(E_y, E_0\right)}{\Omega_\nu\left(E, E_0\right)}\right]\left[\frac{\lambda_\nu(E) - \lambda_\nu(E_0)}{\lambda_\nu(E_y) - \lambda_\nu(E_0)}\right] \\ & \times \exp\left[-x\mathcal{D}_\nu(E, E_y)\right] \mathrm{d}y \to 0.\end{aligned}$$

With the function $\psi_\nu(E_0; E, x)$ in hand, we can obtain the solution to the transport equation (1) for *any* initial spectrum $F_\nu^0(E)$. Indeed, multiplying eq. (A.1) by $F_\nu^0(E_0)$ and integrating over E_0, we have

$$\begin{aligned} F_\nu(E,x) &= \int\limits_0^\infty F_\nu^0(E_0) G_\nu(E_0; E, x)\, \mathrm{d}E_0 \\ &= F_\nu^0(E)\, \mathrm{e}^{-x/\lambda_\nu(E)} + \int\limits_E^\infty F_\nu^0(E_0) \psi_\nu(E_0; E, x)\, \mathrm{e}^{-x/\lambda_\nu(E_0)}\, \frac{\mathrm{d}E_0}{E}. \end{aligned} \tag{A.8}$$

The first term on the right side of eq. (A.8) describes neutrino absorption and the second the neutrino regeneration due to energy loss through the reactions $\nu T \to \nu X$. Eq. (A.8) is in fact equivalent to eq. (2) but, when the function $\psi_\nu(E_0; E, x)$ is known, eq. (A.8) becomes much more convenient for calculations because ψ_ν is independent from the initial spectrum [6] . Due to the mentioned equivalence, we can get a useful representation for the Z factor in terms of the function ψ_ν:

$$Z_\nu(E,x) = \frac{\lambda_\nu(E)}{x} \ln \left[1 + \int\limits_0^1 \eta_\nu(y,E) \psi_\nu(E_y; E, x)\, \mathrm{e}^{-x\mathcal{D}_\nu(E,E_y)}\, \frac{\mathrm{d}y}{1-y} \right]. \tag{A.9}$$

It should be noted that the Z factor calculated in the n-th approximation using the algorithm (5–6) agrees only numerically rather than analytically with that calculated from eq. (A.9), using the iteration algorithm for ψ_ν. In particular, substituting $\psi_\nu = \psi_\nu^{(0)}$ into eq. (A.9) yields

$$Z_\nu(E,x) = \frac{\lambda_\nu(E)}{x} \ln \left[1 + \frac{x Z_\nu^{(1)}(E,x)}{\lambda_\nu(E)} \right] \equiv Z_\nu^{(\mathrm{I})}(E,x),$$

where $Z_\nu^{(1)}(E,x)$ is defined by eq. (8). Thus

$$Z_\nu^{(\mathrm{I})}(E,x) = Z_\nu^{(1)}(E,x) \left[1 - \frac{x Z_\nu^{(1)}(E,x)}{2\lambda_\nu(E)} + \ldots \right] \leq 1.$$

However, the $Z_\nu^{(\mathrm{I})}(E,x)$ can be approximated by $Z_\nu^{(1)}(E,x)$ with very good accuracy, because $x Z_\nu^{(1)}(E,x)/\lambda_\nu(E) \ll 1$ in most cases of interest for neutrino astrophysics.

[6] However, eq. (A.8) has one evident technical drawback. To use it, one must calculate 3-dimensional arrays which are hard to interpolate due to the very strong dependence of ψ_ν and $\mathcal{Z}_\nu$ from their arguments. From this point of view, the algorithm described in sect. 2 is of course simpler.

B Neutrino transport equation with a source function

Here, we briefly show how to take into account the contributions from production of neutrinos through reactions $\nu_\ell T \to \nu_{\ell'} X$ ($\ell \neq \ell'$) or the reaction chains mentioned in the introduction in the case when these may be treated as corrections to the principal solution described in sect. 2 and appendix A. Clearly, the problem reduces to the transport equation (1) with a source function $S_\nu(E,x)$ on the right side. In line with our general approach, we will seek the solution to this equation in the following form[7]

$$F_S(E,x) = \int_0^x S_\nu(E,x') \exp\left[-\int_{x'}^x \frac{1-\mathcal{Z}_\nu(E,x'')}{\lambda_\nu(E)}\,\mathrm{d}x''\right] \mathrm{d}x' \tag{B.1}$$

with $\mathcal{Z}_\nu(E,x)$ a positive function satisfying the equation

$$\mathcal{Z}_\nu(E,x) = \int_0^1 \eta_S(y,E;x)\Phi_\nu(y,E)\,\mathrm{d}y, \tag{B.2}$$

where we introduced

$$\eta_S(y,E;x) = \frac{F_S(E_y,x)}{F_S(E,x)(1-y)}.$$

It is easy to verify that $F_S(E,x) \sim xS_\nu(E,0)$ as $x \to 0$. Therefore,

$$\eta_S(y,E;0) = \frac{S_\nu(E_y,0)}{S_\nu(E,0)(1-y)},$$

and this function is assumed to be finite for any E and y.

The algorithm for the solution to eqs. (B.1), (B.2) is quite obvious: putting $\mathcal{Z}_\nu^{(0)} = 0$ yields

$$F_S^{(0)}(E,x) = \int_0^x S_\nu(E,x-x')\,\mathrm{e}^{-x'/\lambda_\nu(E)}\,\mathrm{d}x',$$

$$\mathcal{Z}_\nu^{(1)}(E,x) = \int_0^1 \eta_S^{(0)}(y,E;x)\Phi_\nu(y,E)\,\mathrm{d}y,$$

[7] We suppose $F_S(E,0) = 0$ as the boundary condition.

etc. The formal question about the finiteness of the involved integrals over y is closely related to the very difficult problem of the asymptotic behavior for the νN inclusive and total cross sections as $E \to \infty$. This problem is beyond the scope of this study, but we can avoid it introducing a cutoff $y_0 = 1 - E/E_0$ (with $E_0 \gg E$) as the upper limit of the integrals. The reason for such a cutoff is in the fact that any physical source function, $S_\nu(E, x)$, must exponentially vanish as $E \to \infty$ (cf. footnote 3).

C Neutrino-nucleon cross sections

According to ref. [13], the inclusive differential cross sections for the reactions $\nu_\mu N \to \mu^- X$ (CC) and $\nu_\mu N \to \nu_\mu X$ (NC), where N is an isoscalar nucleon, in the renormalization-group-improved parton model are of the form ($E \gg M_N$)

$$\frac{\mathrm{d}\sigma_{CC}(y,E)}{\mathrm{d}y} = \frac{2G_F^2 M_N E}{\pi} \int_0^1 \frac{A(\hat{x},Q^2) + (1-y)^2 \overline{B}(\hat{x},Q^2)}{(Q^2/M_W^2+1)^2}\, \mathrm{d}\hat{x},$$

$$\frac{\mathrm{d}\sigma_{NC}(y,E)}{\mathrm{d}y} = \frac{G_F^2 M_N E}{2\pi} \int_0^1 \frac{A_0(\hat{x},Q^2) + (1-y)^2 \overline{B}_0(\hat{x},Q^2)}{(Q^2/M_Z^2+1)^2}\, \mathrm{d}\hat{x}.$$

Here G_F is the Fermi constant, M_N and M_W (M_Z) are the nucleon and W-(Z-) boson masses, respectively, $Q^2 = 2M_N \hat{x} y E$, is the squared invariant momentum transfer between the incident and outgoing lepton, and $\hat{x}$ is the usual Bjorken scaling variable. For the $\overline{\nu}_\mu N$ cross sections, the structure functions A, $\overline{B}$, A_0 and $\overline{B}_0$ in the above formulas should be substituted for the functions $\overline{A}$, B, $\overline{A}_0$ and B_0, respectively. All these are

$$\begin{aligned}
A &= \frac{1}{2}(u_v + d_v + u_s + d_s) + s_s + b_s,\\
\overline{A} &= \frac{1}{2}(u_s + d_s) + s_s + b_s,\\
B &= \frac{1}{2}(u_v + d_v + u_s + d_s) + c_s + t_s,\\
\overline{B} &= \frac{1}{2}(u_s + d_s) + c_s + t_s,\\
A_0 &= \frac{1}{2}\left(L_u^2 + L_d^2\right)(u_v + d_v + u_s + d_s) + \frac{1}{2}\left(R_u^2 + R_d^2\right)(u_s + d_s) + C,\\
\overline{A}_0 &= \frac{1}{2}\left(L_u^2 + L_d^2\right)(u_s + d_s) + \frac{1}{2}\left(R_u^2 + R_d^2\right)(u_v + d_v + u_s + d_s) + C,\\
B_0 &= \frac{1}{2}\left(R_u^2 + R_d^2\right)(u_s + d_s) + \frac{1}{2}\left(L_u^2 + L_d^2\right)(u_v + d_v + u_s + d_s) + C,
\end{aligned}$$

$$\overline{B}_0 = \frac{1}{2}\left(R_u^2 + R_d^2\right)(u_v + d_v + u_s + d_s) + \frac{1}{2}\left(L_u^2 + L_d^2\right)(u_s + d_s) + C,$$

where

$$C = \left(L_d^2 + R_d^2\right)(s_s + b_s) + \left(L_u^2 + R_u^2\right)(c_s + t_s),$$

$R_d = \frac{2}{3}\sin^2\theta_W$, $R_u = -2R_d$, $L_d = R_d - 1$ and $L_u = R_u + 1$ are the chiral couplings, and θ_W is the Weinberg angle. The functions $u_{v,s} = u_{v,s}(\hat{x}, Q^2)$, $d_{v,s} = d_{v,s}(\hat{x}, Q^2)$, etc. are the distributions of corresponding valence (v) and sea (s) quark flavors in a proton. For all constants involved we used the standard values from ref. [16].

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